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ORIGINAL PAPER

Exact solutions of a new Coulomb ring-shaped potential

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Abstract A new Coulomb ring-shaped potential is proposed, which results from adding a potential proportional to$\left(\cos\theta/r^2\sin^2\theta\right)$to a Coulomb potential. The Schrödinger equation with this new model potential is separated into angular and radial components. The exactly wavefunctions and the spectrum equation for bound state are presented by the standard approach.

Keywords Noncentral potential · Coulomb ring-shaped potential · Schrödinger equation · Bound state

1 Introduction

In recent years, noncentral potentials have been the subject of some studies in quantum physics and theoretical chemistry. First, the occurrence of 'accidental' degeneracy and 'hidden' symmetry in the noncentral potentials has caught the attention of many physicists. The fundamental works [1–4] list a variety of noncentral potentials leading to the accidental degeneracy. It turns out that all those noncentral potentials admit the separation of variables in several coordinate systems and possess dynamical symmetries responsible for separability of the Schrödinger equation. As we know, if the noncentral potential maintains separability of the Schrödinger equation, in which case the Schrödinger equation splits into a radial and two angular equations that

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have to be solved. Second, owing to its possible applications in quantum chemistry and nuclear physics to describe ring-shaped molecules like benzene and interactions between deformed pair of nuclei, much work has been devoted to obtain the analytical solutions of the Schrödinger equation with noncentral problems. Several of such noncentral potentials were found, together with their associated closed-form solution of the Schrödinger equation [5–9].

In 1972, an exactly solvable noncentral potential was introduced by Hartmann [10], which can be realized by adding a potential proportional to $(r\sin\theta)^{-2}$to a Coulomb potential. In spherical coordinates (r, θ, φ), the Hartmann potential is defined as

$$V_q(r,\theta) = \eta\sigma^2\left(\frac{2a_0}{r} - q\eta\frac{a_0^2}{r^2\sin^2\theta}\right)\varepsilon_0. \tag{1}$$

Where $a_0 = \hbar^2/Me^2$ and $\varepsilon_0 = -Me^4/2\hbar^2$ represent the Bohr radius and the ground state energy of the Hydrogen atom, respectively, and $\eta$and σ are positive real numbers with values ranging from 1 to 10, and q is a real parameter. When $q = 0$ and $\eta\sigma^2 = Z$, the Hartmann potential reduces to the Coulomb potential. Letting $A = \eta\sigma^2e^2$ and $B = q\hbar^2\eta^2\sigma^2$, the Hartmann potential can be rewritten as

$$V_q(r,\theta) = \frac{-A}{r} + \frac{B}{2Mr^2\sin^2\theta}. \tag{2}$$

The Schrödinger equation with Hartmann potential is separable in both spherical and parabolic coordinates [11,12] and many studies have been focused on this quantum system. For example, the bound state solution and energy spectrum[13–18], the continuous state and phase shifts[19], the 'accidental'degeneracy and 'hidden' symmetry [20,21], the overlap coefficients for two ring-shaped potentials[22]. Besides the standard approach, there are different methods used in the contributions mentioned above and include the Nikiforov-Uvarov method, the SUSY quantum mechanics and shape invariance method, the KS transformation and Laplace transformation, the path integral method [23,24] and the Darboux transformation [25], etc.

Based on the studies mentioned above and our recent works [26,27], we attempt to propose a new noncentral potential, or another Coulomb ring-shaped potential given by

$$V(r,\theta) = \frac{-A}{r} + \frac{B\cos\theta}{2Mr^2\sin^2\theta}. \tag{3}$$

The purpose of this work is to investigate the Schrödinger equation with this noncentral potential in the framework of a standard approach.

This work is organized as follows: In Sect. 2 the Schrödinger equation with this potential is separated into a radial and two angular equations, the boundary conditions for both the angular and radial equations are discussed. In Sect. 3 the exactly complete solutions of this quantum system are obtained. Finally, some concluding remarks are given in Sect. 4.

2 The Schrödinger equation with the Columbic ring-shaped potential

Throughout this work the natural units ($\hbar = M = 1$)are employed for simplicity. The Schrödinger equation with this new proposed potential (3) is given by

$$\left\{-\frac{1}{2r^2}\left[\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r}+\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta}+\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right]\right.$$
$$\left.+V(r,\theta)-E\right\}\psi(r,\theta,\varphi)=0. \tag{4}$$

Where E is the energy. In spherical coordinate, one may select the wave function

$$\psi(r,\theta,\phi)=r^{-1}u(r)\,H(\theta)\,K(\phi). \tag{5}$$

Substituting Eq. 5 into Eq. 4 leads to a set of second-order differential equations

$$\frac{1}{\sin\theta}\frac{d}{\mathrm{d}\theta}\left(\sin\theta\frac{d}{\mathrm{d}\theta}\right)H(\theta)-\left[\lambda-\frac{m^2+2B\cos\theta}{\cos^2\theta}\right]H(\theta)=0. \tag{6}$$

$$\frac{d^2}{\mathrm{d}r^2}u(r)+\left[2E+\frac{2A}{r}-\frac{\lambda}{r^2}\right]u(r)=0. \tag{7}$$

$$\frac{d^2K(\phi)}{\mathrm{d}\phi^2}+m^2K(\phi)=0. \tag{8}$$

Where m^2 and λ are two separation constants. The boundary conditions for Eq. 6 require that $H(0)$ and $H(\pi)$ are taken as a finite value. However, the boundary conditions for Eq. 7 require $u(0) = 0$ and the square-integrability of $u(r)$ on $(0, \infty)$, which implies $u(\infty) = 0$ for the bound states. The periodic boundary condition of Eq. 8 is given by $K(\phi + 2\pi) = K(\phi)$. Then the solution of Eq. 8 can be obtained immediately as

$$K_m(\phi)=\frac{1}{\sqrt{2\pi}}\exp(im\phi),\quad m=0,\pm1,\pm2\ldots \tag{9}$$

3 The normalized wave functions and the spectrum of bound states

First, let us consider the θ-dependent Eq. 6. By introducing a new variable [28]

$$x=\frac{1}{2}(1+\cos\theta)=\cos^2\frac{\theta}{2}. \tag{10}$$

and defining two positive numbers

$$p=\frac{1}{2}\left|m^2-2B\right|^{\frac{1}{2}}\geq0,\quad q=\frac{1}{2}\left|m^2+2B\right|^{\frac{1}{2}}\geq0. \tag{11}$$

Equation 6 could be rearranged as

$$x(1-x)\frac{d}{\mathrm{d}x}x(1-x)\frac{d}{\mathrm{d}x}H(x) + \left[\lambda x(1-x) - p^2 - \left(q^2 - p^2\right)x\right]H(x) = 0. \tag{12}$$

The physically acceptable solution of Eq. 12 could be expressed as

$$H(x) = x^p(1-x)^q g(x). \tag{13}$$

Substitution of Eq. 13 into Eq. 12, we get

$$x(1-x)\frac{d^2}{\mathrm{d}x^2}g(x) + [k-(i+j+1)x]\frac{d}{\mathrm{d}x}g(x) - (i\times j)\,g(x) = 0. \tag{14}$$

Equation (14) is a hypergeometric equation and its solution is the hypergeometric function

$$g(x) = F(i,j,k,x). \tag{15}$$

In which

$$\begin{cases} i = p+q-\ell, \\ j = p+q+1+\ell, \\ k = 2p+1. \end{cases} \tag{16}$$

$$\lambda = \ell(\ell+1), \quad \ell = 0,1,2\ldots. \tag{17}$$

However, the boundary conditions satisfied by the θ-angular wave function demands that the hypergeometric function must be terminated as a polynomial, this demands

$$i = p+q-\ell = -n_3, \qquad (n_3 = 0,1,2\ldots) \tag{18}$$

$$j = p+q+1+\ell = -n_3. \quad (n_3 = 0,1,2\ldots) \tag{19}$$

Although the exchange between parameters i and j are equivalent for the hypergeometric function, but only Eq. 18 is satisfied since p and q are two positive number. Where $p+q$ do not need to be integers, Just the difference between $p+q$ and ℓ must be integers. This is to say $\ell = p+q+n_3$ is an integer. In fact, ℓ is the total angular momentum number. Finally, we get the normalized solutions of θ- dependent Eq. 6 as

$$H_{n_3}(\theta) = N_{n_3}\cos^{2p}\frac{\theta}{2}\sin^{2q}\frac{\theta}{2}F\left(-n_3, p+q+1+\ell, 2p+1, \cos^2\frac{\theta}{2}\right), \tag{20}$$

where $N_{n_3}$is the normalization constant. By using the following relations [29]

$$\begin{cases} j_n(\alpha,\gamma,x) = F(-n,\alpha+n,\gamma,x), \\ \int\limits_0^1 x^{\gamma-1}(1-x)^{\alpha-\gamma} j_n(\alpha,\gamma,x) j_k(\alpha,\gamma,x)\mathrm{d}x = \frac{\Gamma^2(\gamma)\Gamma(\alpha+n-\gamma+1)}{\Gamma(\alpha+n)\Gamma(\gamma+n)}\frac{n!}{(\alpha+2n)}\delta_{nk}, \end{cases} \tag{21}$$

and the orthogonality of the angular wave functions

$$\int\limits_{-1}^{1} [H_n(x)]^2 dx = 1, \tag{22}$$

the normalization constant is obtained as

$$N_{n_3} = \frac{1}{(2p)!}\sqrt{\frac{(2p+2q+n_3)!(2p+n_3)!\,(2p+2q+2n_3+1)}{2n_3!(2q+n_3)!}}. \tag{23}$$

For the radial Eq. 7, another variable $\rho = sr$ is introduced. By setting

$$2E = -\frac{s^2}{4}, \quad t = \frac{2A}{s}. \tag{24}$$

Equation (7) can be rearranged as

$$\frac{d^2}{\mathrm{d}\rho^2}u(\rho) + \left[-\frac{1}{4} + \frac{t}{\rho} - \frac{\ell(\ell+1)}{\rho^2}\right]u(\rho) = 0. \tag{25}$$

The radial Eq. 25 has an irregular singularity at $\rho = \infty$. Furthermore, it has a singularity at $\rho = 0$. Therefore, it is reasonable to set

$$u(\rho) = \rho^{\ell+1}e^{-\frac{\rho}{2}}f(\rho). \tag{26}$$

Substituting Eq. 26 into Eq. 25, we obtain

$$\frac{d^2}{\mathrm{d}\rho^2}f(\rho) + [2(\ell+1)-\rho]\frac{d}{\mathrm{d}\rho}f(\rho) + [t-(\ell+1)]f(\rho) = 0. \tag{27}$$

Equation (27) is a confluent hypergeometric equation, and its solution is given by the confluent hypergeometric function

$$f(\rho) = F(\alpha,\gamma,\rho). \tag{28}$$

In which

$$\alpha = (\ell+1) - t, \quad \gamma = 2(\ell+1). \tag{29}$$

When $\rho \to \infty$, the confluent hypergeometric function $F(\alpha,\gamma,\rho)$ behaves as exp(ρ), where $u(\rho)$ is exponentially divergent. The regularity conditions imply that bound states that exist only are $\alpha = -n_r$ ($n_r = 0, 1, 2, \ldots.$), that is

$$t = n_r + \ell + 1 = n. \quad (n = 0, 1, 2 \cdots) \tag{30}$$

where n_r is the number of nodes of the radial wave functions and n is the usual principal quantum number, respectively. Therefore, we obtain the energy equation

$$E = -\frac{\left(\eta\sigma^2 e^2\right)^2}{2n^2}. \tag{31}$$

Equation (31) is similar as those of the Hartmann potential and other Coulomb ring-shaped potentials, this is to say that the ring-shaped potentials have no influence on the energy spectrum. The corresponding normalized radial wave function is expressed as

$$u_{n_r\ell}(r) = N_{n_r\ell}(sr)^{\ell+1}\exp\left(-\frac{1}{2}sr\right)F(-n_r, 2\ell+2, sr). \tag{32}$$

By using the following known relations

$$L_n^\mu(x) = \frac{\Gamma(n+\mu+1)}{n!\Gamma(\mu+1)}F(-n, \mu+1, x), \tag{33}$$

$$\int_0^\infty x^\mu e^{-x} L_n^\mu(x) L_k^\mu(x)\mathrm{d}x = \frac{\Gamma(n+\mu+1)}{n!}\delta_{nk}, \tag{34}$$

$$(n+1)\,L_{n+1}^\mu(x) + (x-\mu-2n-1)\,L_n^\mu(x) + (\mu+n)\,L_{n-1}^\mu(x) = 0, \tag{35}$$

and the orthogonality of radial wave functions,

$$\int_0^\infty [R_n(r)]^2 r^2 \mathrm{d}r = 1, \left[R_n(r) = r^{-1}u_n(r)\right], \tag{36}$$

the normalized radial wave functions can be written as

$$u_{n\ell}(r) = \sqrt{\frac{s\,(n-\ell-1)!}{2n\Gamma(n+\ell+1)}}\left(\sqrt{-8E}r\right)^{\ell+1}\exp\left(-\sqrt{-2E}r\right)L_{n'}^\mu\left(\sqrt{-8E}r\right). \tag{37}$$

Finally, we get the normalized wave functions of this quantum system as

$$\psi(r,\theta,\varphi) = \frac{1}{\sqrt{2\pi}}r^{-1}u_{n\ell}(r)\,H_{n_3}(\theta)\exp(im\varphi), \quad (m = \pm 0, 1, 2\ldots). \tag{38}$$

4 Conclusion and remarks

We have proposed a new exactly solvable noncentral potential, or another Coulomb ring-shaped potential and solved the Schrödinger equation with this potential by the standard method analytically. It is shown that in spherical coordinate, the Schrödinger equation with this noncentral potential could be separated into the angular and radial components. The radial equation is similar as that of the Hartmann potential and the normalized radial wave functions are given by the generalized-Laguerre polynomials. The angular component is different to that of the Hartmann potential and is satisfies a hypergeometric equation, the normalized angular wave functions are expressed in terms of the hypergeometric functions. The spectrum equation for bound state is derived also.

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References

1. A.A. Makarov, J.A. Smorodinsky, K.h. Valiev, P. Winternitz, Nuovo Cimento A **52**, 1061 (1967)
2. M. Kibler, L.G. Mardoyan, G.S. Pogosyan, Int. J. Quant. Chem. **52**, 1301 (1994)
3. M. Kibler, G.H. Lamot, P. Winternitz, Int. J. Quant. Chem. **43**, 625 (1992)
4. C. Quesne, J. Phys. A: Math. Gen. **21**, 3093 (1988)
5. R. Dutt, A. Gangopadhyaya, U. Sukhatme, Am. J. Phys. **65**, 400 (1997)
6. A. Khare, R.K. Bhaduri, Am. J. Phys. **62**, 1008 (1994)
7. S.H. Dong, C.Y. Chen, M. Lozada-Cassou, Int. J. Quant. Chem. **105**, 453 (2005)
8. G.A. Kerimov, J. Phys. A: Math. Gen. **40**, 7297 (2007)
9. C. Berkdemir, J. Math. Chem. **46**, 139 (2009)
10. H. Hartmann, Theor. Chim. Acta **24**, 201 (1972)
11. H. Hartmann, R. Schuck, J. Radtke, Theor. Chim. Acta **46**, 1 (1976)
12. H. Hartmann, D. Schuch, Int. J. Quant. Chem. **18**, 125 (1980)
13. M. Kibler, T. Negadi, Int. J. Quant. Them. **26**, 405 (1984)
14. M. Kibler, T. Negadi, Theor. Chim. Acta **66**, 31 (1984)
15. G.G. Blado, Int. J. Quant. Chem. **58**, 431 (1996)
16. G.G. Blado, Theor. Chim. Acta **94**, 53 (1996)
17. C.Y. Chen, Phys. Lett. A **339**, 283 (2005)
18. M. Aktas, R. Sever, J. Math. Chem. **37**, 139 (2005)
19. C.Y. Chen, F.L. Lu, D.S. Sun, Phys. Lett. A **329**, 420 (2004)
20. M. Kibler, P. Winternitz, J. Phys. A: Math. Gen. **20**, 4097 (1992)
21. Y.I. Granovskii, A.S. Zhedanov, I.M. Lutzenko, J. Phys. A: Math. Gen. **24**, 3887 (1991)
22. A.S. Zhedanov, J. Phys. A: Math. Gen. **26**, 4633 (1993)
23. I. Sökmen, Phys. Lett. A **115**, 249 (1986)
24. B.P. Mandal, Int. J. Mod. Phys. A **15**, 1225 (2000)
25. A. Schulze-Halberg, E. Zamora-Gallardo, J.J. Pena, Int. J. Quant. Chem. **109**, 1464 (2009)
26. M.C. Zhang, Acta. Phys. Sin. **58**, 712 (2009)
27. M.C. Zhang, Cent. Eur. J. Phys. **7**, 768 (2009)
28. S. Flügge, *Practical Quantum Mechanics* (Springer, Berlin, 1971)
29. Z.X. Wang, D.R. Guo, *Special Functions* (Word Scientific, Singapore, 1989)